Ambiguities and Symmetry Relations in a Free Fermion Model

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ABSTRACT

We present a systematic study of one, two and three point functions of vector, axial-vector, scalar and pseudoscalar densities constructed in a free fermion model. The divergent content of the amplitudes are left in the form of (external momenta independent) 4-D integrals for which an appropriate regulating function is only implicitly assumed, and the integrals are not evaluated at any step of the calculation. The ambiguities and Symmetries Violations, in all cases, are shown to be associated with coefficients involving three relations between divergent integrals of the same degree of divergence. Setting these coefficients to zero is mandatory, e.g., for preserving gauge symmetry in QED. The implications for the ambiguities and symmetry violations are investigated. The results emerging from this alternative approach allow us to conclude that the traditional method used to establish the triangular anomalies could be questionable.

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1 Introduction

Quantum Field Theory is nowadays our main tool for the investigation of the elementary particle interactions. This is also due to the fact that it allows us to study the consequences of symmetries assumed relevant, our fundamental working hypothesis. Since this is seldom possible in an exact way, perturbative techniques become most relevant. Unfortunately this type of solution is plagued with mathematical problems coming from divergent integrals. We are therefore forced to have recourse of auxiliary techniques based on regularization schemes in order to extract the physical content of calculated amplitudes. Regularization schemes invariably modify the amplitudes, as dictated by Feynman rules, and consequently provide an interpretation for the mathematical indefinitions in the problem. Frequently the final results become regularization-scheme dependent and may not reflect the full dynamical content of the underlying theory. In this context one of the main issues is the question of ambiguities associated with arbitrary momentum routing in loops and that of symmetry relations [1]. Many of these fundamental questions have been adequately solved after the construction of Dimensional Regularization [2]. There is however one important issue which, due to the mathematical limitation of the technique, could not be handled, namely the pseudoamplitudes. Perhaps this is the reason why even nowadays the treatment given to such amplitudes have recourse of methods completely discarded in all cases where D.R. applies. We refer Specifically to the problem of triangle anomalies, which has been discussed in the pioneer work of Gerstein and Jackiw [1]. They proposed that surface terms, which are in turn equivalent to admit an intrinsic ambiguity of the divergent amplitudes, are responsible for the existence of symmetry violations in several of such amplitudes. This point of view is completely discarded for all other cases since it represents the violation of the most basic symmetry principle present in the original theory, which is the space-time homogeneity (translational invariance). Despite of this, even in modern text books [3], anomalies are presented as associated with such ambiguities. On the other hand anomalies are shown to emerge even in theories destituted of divergences and predicted by general arguments in QFT. It seems to have no doubt about their existence. Therefore it is reasonable to believe that it must be possible to establish their existence without having to appeal to ambiguities, even when divergent amplitudes are involved.

This work is devoted to question the origin of anomalies as they are usually presented in the literature and providing a universal point of view to treat divergencies in the sense that all amplitudes are handled by the same prescription. For this purpose we appeal to a technique to manipulate and calculate divergent integrals which has recently been proposed [4] and successfully applied to the problem of ambiguities in the context of the gauged NJL model [5], [6] solving those crucial question raised in the literature [7]. This method so far has proved to be an adequate tool to treat all questions related with divergencies from the renormalization of standard theories like QED [8], [9] and for the calculation of renormalization group coefficients [10]. In this context we turn in the present work to perhaps the historically most relevant question that involves the association between the violation of symmetry relations and ambiguities. The advantage of the method adopted here is that it allows an immediate connection with other current approaches. In particular we can use the consistent results obtained by D.R. where it applies as element of the analysis for the search of an universal interpretation. It is most desirable to obtain the anomalies in a natural way within a context which treats all anomalous and nonanomalous amplitudes according to the same scheme. This is better done in two steps. First (as in the present work) one must check wether the current way as followed in ref.[1] for calculating e.g. the famous axial anomaly remains appropriate when used to investigate other symmetry relations. In other words, anomalies must also emerge in a treatment designed to consistently handle all other amplitudes, i.e., where ambiguities are necessary absent. We show as a consequence of a thorough analysis that D.R. and the prescription of ref.[1] are conflicting, i.e., cannot be mapped by a universal interpretation. An important result of thus analysis is that when we demand complete mapping of the present results (in 4-D) with those of D.R. we obtain a set of conditions which are not compatible with any symmetry violation even in the anomalous cases. This indicates that the adopted procedure which makes use only of two point functions is not completely consistent. In view of the present results we conclude that final and decisive statements about three point function symmetry relations can only be made after an explicit calculation of these amplitudes. This is the second step. It is possible to show that anomalies emerge in a natural way with the correct value [11] without having to admit ambiguities and within an interpretation which is universal in a sense that consistent results are obtained for all amplitudes in the same way [4].

As discussed before we consider the free fermion model of ref.[1] discarding however internal symmetries, which are irrelevant for our purposes. In section 2 we define the model, notation and relevant Ward Identities. In section 3 we briefly establish the operational strategy for the manipulation and calculation of divergent amplitudes. In section

4 a study of ambiguities is presented in one and two point functions calculated explicitly with arbitrary momentum routing. We also explicitate the ambiguous terms of the three point functions. In section 5 we reproduce (from our results) those of ref.[1]. In section 6 we discuss Ward Identities. The final remarks are contained in section 7.

2 Definitions, Notation and Current Algebra Results for the Model

We start introducing the notation to be used and defining the quantities we will be concerned with for the rest of the work. We closely follow ref.[1] with which we compare our results.

Let us consider a spin 1/2, mass m free fermion model. There will be therefore a massive field which obeys Dirac's equation and with which we can construct currents $j_i(x)$ defined by

$$j_i(x) = \bar{\psi}(x)\Gamma_i\psi(x), \tag{1}$$

where Γ_i are the Dirac matrices responsible for the transformation of currents:

$$\Gamma_i = [S(x); P(x); V_\mu(x); A_\mu(x)] = [\hat{1}; \gamma_5; \gamma_\mu; i\gamma_\mu\gamma_5] \tag{2}$$

characterizing the scalar, pseudo-scalar, vector and axial densities. An important property in this model is the value of the four divergencies

$$\begin{cases} \partial_\mu V_\mu(x) = 0 \\ \partial_\mu A_\mu(x) = 2mP(x), \end{cases} \tag{3}$$

and their commutation relations at equal times. With the above definitions and the fermionic propagator

$$iS_F(p) = \frac{i}{\not{p} - m}, \tag{4}$$

it is possible to construct n-point functions, which we define in the same way as in ref.[1] as follows

- One point functions:

$$T^i(k_1, m) = \int \frac{d^4k}{(2\pi)^4} Tr\left\{\Gamma_i \frac{1}{[(\not{k} + \not{k}_1) - m]}\right\}, \tag{5}$$

- Two point functions:

$$T^{ij}(k_1,m;k_2,m)=\int\frac{d^4k}{(2\pi)^4}Tr\left\{\Gamma_i\frac{1}{[(\not{k}+\not{k}_1)-m]}\Gamma_j\frac{1}{[(\not{k}+\not{k}_2)-m]}\right\} \tag{6}$$

- Three point functions:

$$\begin{aligned}&T^{lij}(k_1,m;k_2,m;k_3,m)=\\&\quad\int\frac{d^4k}{(2\pi)^4}Tr\left\{\Gamma_l\frac{1}{[(\not{k}+\not{k}_3)-m]}\Gamma_i\frac{1}{[(\not{k}+\not{k}_1)-m]}\Gamma_j\frac{1}{[(\not{k}+\not{k}_2)-m]}\right\},\end{aligned} \tag{7}$$

  and so on. Here the k_1, k_2 and k_3 represent the arbitrary choice for the internal momenta of the loop. Energy momentum conservation only requires these quantities must be related with the external momenta, e.g., in the three point functions we have: $k_3-k_1=p$, $k_1-k_2=p'$ and $k_3-k_2=q$.

In our notation the vertex to the left is assumed to be connected with the "initial state" with external momentum q and the vertex operator Γ_l. The other two vertices correspond to "final states" with external momenta p and p' and the vertex operators Γ_i e Γ_j respectively. Besides the upper indices are associated (see figure), in the order they appear, with the respective Lorentz indices in the same order, whenever is the case. As an example if $\Gamma_l=\gamma_\lambda$; $\Gamma_i=i\gamma_\mu\gamma_5$ and $\gamma_j=\gamma_5$ we have

$$T^{VAP}_{\lambda\mu}(k_1,m;k_2,m;k_3,m). \tag{8}$$

This notation emphasizes the masses and momenta carried by the internal lines and characterizes each diagram completely. In particular this means that in the case we have one particle in the initial state and two on the final one, symmetrization in the final states will be required. For example for the process $S\to VV$ we define the corresponding amplitude as

$$T^{S\to VV}_{\mu\nu}=T^{SVV}_{\mu\nu}(k_1,m;k_2,m;k_3,m)+T^{SVV}_{\nu\mu}(l_1,m;l_2,m;l_3,m) \tag{9}$$

The first term represents the direct channel and the second the crossed channel.

There are integral representations for the functions defined above with the Fourier transform of the currents

$$<j_1(q)j_2(-q)>\equiv\int e^{-ipx}d^4x<0|T(j_1(x)j_2(0)|0> \tag{10}$$

in the case of two point functions and

$$< j_1(p) j_2(p') j_3(q) > \equiv \int e^{-ipx} e^{-ip'y} d^4x d^4y < 0|T(j_1(x) j_2(y) j_3(0))|0 > \tag{11}$$

for the three point functions. With these elements and the standard methods of current algebra [3], [11] one can establish relations among the n-point functions, i.e., Ward identities. For the simple model in question such symmetry relations are reduced to the conservation of the vector current and the well known proportionality of the divergent of the axial current and the pseudoscalar one. It is important to remark that such relations are exact and should be satisfied despite the divergent character of up to n-point functions.

This brief discussion summarizes the dilemma one has to face when calculating divergent amplitudes: to maitain their properties after the calculation. It is our purpose to investigate under which conditions it is possible to evaluate these amplitudes and to get consistent results in what concerns ambiguities and symmetry relations.

The Ward identities we should verify are the following

- *One point functions*

$$T_\mu^V(l.m) = 0 \tag{12}$$

- *Two point functions*

$$(k_1 - k_2)_\mu T_\mu^{VS}(k_1, m; k_2, m) = 0 \tag{13}$$

$$(k_1 - k_2)_\mu T_{\mu\nu}^{VV}(k_1, m; k_2, m) = 0 \tag{14}$$

$$(k_1 - k_2)_\nu T_{\mu\nu}^{VV}(k_1, m; k_2, m) = 0 \tag{15}$$

$$(k_1 - k_2)_\mu T_\mu^{AP}(k_1, m; k_2, m) = -2mi T^{PP}(k_1, m; k_2, m) \tag{16}$$

$$(k_1 - k_2)_\mu T_{\mu\nu}^{AV}(k_1, m; k_2, m) = -2mi T_\nu^{PV}(k_1, m; k_2, m) \tag{17}$$

$$(k_1 - k_2)_\nu T^{AV}_{\mu\nu}(k_1, m; k_2, m) = 0 \tag{18}$$

$$(k_1 - k_2)_\mu T^{AA}_{\mu\nu}(k_1, m; k_2, m) = -2miT^{PA}_\nu(k_1, m; k_2, m) \tag{19}$$

$$(k_1 - k_2)_\nu (k_1 - k_2)_\mu T^{AA}_{\mu\nu}(k_1, m; k_2, m) = (2m)^2 T^{PP}(k_1, m; k_2, m) \tag{20}$$

- *Three point functions*

$$q_\lambda T^{V\to SS}_\lambda = 0 \tag{21}$$

$$q_\lambda T^{V\to PP}_\lambda = 0 \tag{22}$$

$$q_\lambda T^{A\to SP}_\lambda = -2miT^{P\to SP} \tag{23}$$

$$p'_\nu T^{S\to VV}_{\mu\nu} = 0 \tag{24}$$

$$p_\mu T^{S\to AA}_{\mu\nu} = 2miT^{S\to PA}_\nu \tag{25}$$

$$p'_\nu T^{S\to AA}_{\mu\nu} = 2miT^{S\to AP}_\mu \tag{26}$$

$$p_\mu T^{P\to AV}_{\mu\nu} = 2miT^{P\to PV}_\nu \tag{27}$$

$$p'_\nu T^{P\to AV}_{\mu\nu} = 0 \tag{28}$$

$$p_\mu T^{P\to VV}_{\mu\nu} = 0 \tag{29}$$

$$p'_\nu T^{P\to VV}_{\mu\nu} = 0 \tag{30}$$

$$p_\mu T^{S\to AV}_{\mu\nu} = 2miT^{S\to PV}_{\nu} \tag{31}$$

$$p'_\nu T^{S\to AV}_{\mu\nu} = 0 \tag{32}$$

$$p_\mu T^{P\to AA}_{\mu\nu} = 2miT^{P\to PA}_{\nu} \tag{33}$$

$$p'_\nu T^{P\to AA}_{\mu\nu} = 2miT^{P\to AP}_{\mu} \tag{34}$$

$$q_\lambda T^{A\to VV}_{\lambda\mu\nu} = -2miT^{P\to VV}_{\mu\nu} \tag{35}$$

$$p_\mu T^{A\to VV}_{\lambda\mu\nu} = 0 \tag{36}$$

$$p'_\nu T^{A\to VV}_{\lambda\mu\nu} = 0 \tag{37}$$

$$q_\lambda T^{A\to AA}_{\lambda\mu\nu} = -2miT^{P\to AA}_{\mu\nu} \tag{38}$$

$$p_\mu T^{A\to AA}_{\lambda\mu\nu} = 2miT^{A\to PA}_{\lambda\nu} \tag{39}$$

$$p'_\nu T^{A\to AA}_{\lambda\mu\nu} = 2miT^{A\to PA}_{\lambda\mu} \tag{40}$$

$$q_\lambda T^{V\to VV}_{\lambda\mu\nu} = 0 \tag{41}$$

$$p_\mu T^{V\to VV}_{\lambda\mu\nu} = 0 \tag{42}$$

$$p'_\nu T^{V\to VV}_{\lambda\mu\nu} = 0 \tag{43}$$

$$q_\lambda T^{V\to AA}_{\lambda\mu\nu} = 0 \tag{44}$$

$$p_\mu T^{V\to AA}_{\lambda\mu\nu} = 2miT^{V\to PA}_{\lambda\nu} \tag{45}$$

$$p'_\nu T^{V\to AA}_{\lambda\mu\nu} = 2miT^{V\to AP}_{\lambda\mu} \tag{46}$$

3 Proposed Strategy to Manipulate Divergent Integrals

For the following calculations we will handle divergent amplitudes without explicit use of a specific regulating function. However to assume an implicit regulator in the integrands satisfying some properties;

$$\int d^4k f(k) \longrightarrow \int d^4k f(k) G_\Lambda(k^2) \tag{47}$$

such that the connection limit exists, i.e.,

$$\lim_{\Lambda\to\infty} G_\Lambda(k^2) = 1, \tag{48}$$

and, consequently, finite contributions can be integrated without restrictions. On general grounds the regulating function should be even in k, and assuming an implicit regulating function and performing the Dirac traces we identify the set of divergent integrals to be dealt with. Next we manipulate each integral by using identities at the level of the integrand such that all the dependence on external momenta must be contained by the finite terms. For the present model all the divergent content of the amplitudes will be restricted to two integrals which we define as

$$I_{quad}(m^2) = \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)} \tag{49}$$

and

$$I_{log}(m^2) = \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2}. \tag{50}$$

At the end the results obtained will not be committed to any specific regularization scheme. The current reported results with well known schemes can easily be obtained from our expressions. The main advantage in using the new precept is that we will be able to realize which are the conditions to be satisfied by any regularization prescription in order to avoid ambiguities and undesirable symmetry violations.

4 Ambiguities

The integral representation, eq.(6), for two point functions indicates, by power counting, that it is quadratically divergent. Also, eq.(7) indicates that the three point functions are linearly divergent. In ref.[1] the adopted point of view is that, even after the finite content of the integrals are extracted, they remain ambiguous quantities. The reason for this is that the energy-momentum conservation relations do not uniquely specify the internal momenta in the loop. It is possible to make different choices for the internal momentum label. Such choices can only be equivalent if shifts in the integration variable is allowed, which is not the case for linearly and quadratically divergent integrals [1], [3]. Following we explicitly evaluate the up to three point Green's functions in detail in order to illustrate our strategy for the manipulation and calculation of the divergent objects. Special attention will be paid for the question of ambiguities showing how they can be systematically eliminated in the present calculation.

4.1 One Point Functions

We start by considering the one point vector amplitude T^V_μ. According to the standard procedure this function contains one fermionic propagator and the vertex operator $\Gamma_i = \gamma_\mu$. It is defined as

$$T^V_\mu = \int \frac{d^4k}{(2\pi)^4} Tr\left\{\gamma_\mu \frac{1}{(\not{k} + \not{k}_1) - m}\right\}, \tag{51}$$

after the trace is taken we get

$$T^V_\mu = 4\left\{\int \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{(k+k_1)^2 - m^2} + k_{1\mu} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k+k_1)^2 - m^2}\right\}. \tag{52}$$

¿From the above equation we see that we get two divergent integrals, one of cubic and the other of quadratic divergences. Following our strategy we admit the presence of a implicit regulator as discussed before. In order to indicate its presence we use the subscript Λ in the integral and proceed to the necessary manipulations of the integrand.

The cubically divergent integral is written as

$$
\begin{aligned}
\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{k_\mu}{(k+k_1)^2-m^2} &= -k_{1\nu}\left\{\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{2k_\mu k_\nu}{(k^2-m^2)^2}\right\} \\
&\quad + k_{1\nu}k_{1\alpha}k_{1\beta}\left\{\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{4g_{\alpha\beta}k_\mu k_\nu}{(k^2-m^2)^3} - \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{8k_\alpha k_\beta k_\mu k_\nu}{(k^2-m^2)^4}\right\} \\
&\quad - \left\{\int \frac{d^4k}{(2\pi)^4}\frac{6k_1^4 k_{1\nu}k_\mu k_\nu}{(k^2-m^2)^4} + \int \frac{d^4k}{(2\pi)^4}\frac{(k_1^2+2k_1\cdot k)^4 k_\mu}{(k^2-m^2)^4[(k+k_1)^2-m^2]}\right\}, \quad (53)
\end{aligned}
$$

with we odd integrals vanishing. The two last integrals are finite and can therefore be integrated without restrictions cancelling each other. The remaining divergent integrals, easily identified in eq.(53), are left in integral form. For the quadratically divergent integral, performing the same kind of algebraic manipulations at the level of the integrand results

$$
\begin{aligned}
\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{(k+k_1)^2-m^2} &= \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)} \\
&\quad + k_{1\mu}k_{1\nu}\left\{\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{4k_\mu k_\nu}{(k^2-m^2)^3} - \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{g_{\mu\nu}}{(k^2-m^2)^2}\right\} \\
&\quad + \left\{\int \frac{d^4k}{(2\pi)^4}\frac{k_1^4}{(k^2-m^2)^3} + \int \frac{d^4k}{(2\pi)^4}\frac{(k_1^2+2k_1\cdot k)^3}{(k^2-m^2)^3[(k+k_1)^2-m^2]}\right\} \quad (54)
\end{aligned}
$$

where, again, the finite parts of the integral cancel out. Collecting now all the results for T_μ^V we obtain

$$
\begin{aligned}
T_\mu^V &= 4\left\{-k_{1\beta}\nabla_{\beta\mu} - \frac{k_{1\beta}k_{1\alpha}k_{1\nu}}{3}\Box_{\alpha\beta\mu\nu}\right. \\
&\quad \left. - \frac{k_{1\alpha}k_{1\beta}k_{1\mu}}{3}\triangle_{\beta\alpha} + \frac{k_1^2 k_{1\nu}}{3}\triangle_{\nu\mu} + k_{1\mu}k_{1\alpha}k_{1\nu}\triangle_{\alpha\nu}\right\} \quad (55)
\end{aligned}
$$

where we have introduced a set of differences between divergent integrals of the same degree of divergence in the form

$$
\begin{aligned}
\Box_{\alpha\beta\mu\nu} &= \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{24k_\mu k_\nu k_\alpha k_\beta}{(k^2-m^2)^4} - g_{\alpha\beta}\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{4k_\mu k_\nu}{(k^2-m^2)^3} \\
&\quad - g_{\alpha\mu}\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{4k_\beta k_\nu}{(k^2-m^2)^3} - g_{\alpha\nu}\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{4k_\beta k_\mu}{(k^2-m^2)^3}, \quad (56) \\
\nabla_{\mu\nu} &= \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{2k_\mu k_\nu}{(k^2-m^2)^2} - \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{g_{\mu\nu}}{(k^2-m^2)}, \quad (57) \\
\triangle_{\mu\nu} &= \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{4k_\mu k_\nu}{(k^2-m^2)^3} - \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{g_{\mu\nu}}{(k^2-m^2)^2}. \quad (58)
\end{aligned}
$$

These relations appearing in all T_μ^V terms, are explicitly ambiguous due the presence of the momentum k_1 in the coefficients.

The above procedure clearly states our strategy for handling the Feynmam diagrams. As we anticipated, we make use only of algebraic manipulations at the integrand level and the irrestrict integration of finite contributions. In these the subscript Λ has been removed since in the connection limit the integrals can be performed without restrictions.

Let us now treat the remaining one point functions within the same calculational scheme. Let us take following scalar function

$$T^S = \int \frac{d^4k}{(2\pi)^4} Tr \left\{ \hat{1} \frac{1}{(\not{k} + \not{k}_1) - m} \right\} \tag{59}$$

which, after Dirac's trace yields

$$T^S = 4m \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k + k_1)^2 - m^2}. \tag{60}$$

The quadratically divergent integral has already been discussed in the previous example and can be written as

$$T^S = 4m \left\{ I_{quad}(m^2) + k_{1\beta} k_{1\alpha} \triangle_{\beta\alpha} \right\}. \tag{61}$$

In the above result we verify that the amplitude T^S is expressed in terms of two divergent objects, $I_{quad}(m^2)$ and $\triangle_{\beta\alpha}$, the latter again related to the arbitrary momentum label k_1. For the moment we only comment that the two calculated amplitudes are <u>potentially</u> ambiguous due to the presence of such terms in the final expression.

It is a simple matter to check that the other one point functions identically T_μ^A and T^P vanish due the properties of the trace of the corresponding Dirac matrices.

4.2 Two Point Functions

Let us calculate the two point functions within the same scheme. We start by considering the simplest of the two point functions with two scalar vertices. Using Feynman rules with two propagators and appropriate vertices we write

$$T^{SS} = \int \frac{d^4k}{(2\pi)^4} Tr \left\{ \hat{1} \frac{1}{(\not{k} + \not{k}_1) - m} \hat{1} \frac{1}{(\not{k} + \not{k}_2) - m} \right\}, \tag{62}$$

where k_1 and k_2 stand for the arbitrary shifts in the internal momentum routing, as before, and will systematically locate ambiguities, wherever the case may be. Taking Dirac's trace we have

$$T^{SS} = 2\left\{\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k+k_1)^2-m^2} + \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k+k_2)^2-m^2}\right.$$
$$\left. +[4m^2-(k_1-k_2)^2]\int \frac{d^4k}{(2\pi)^4}\frac{1}{[(k+k_1)^2-m^2][(k+k_2)^2-m^2]}\right\} \qquad (63)$$

where we have identified the quadratically divergent integrals which already appeared in previous calculations. As for the second, logarithmically divergent we apply the appropriate manipulations to cast it into the form

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4}\frac{1}{[(k+k_1)^2-m^2][(k+k_2)^2-m^2]} &= \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)^2} \\ &-\int \frac{d^4k}{(2\pi)^4}\frac{(k_1^2+2k_1\cdot k)}{[(k^2-m^2)^2][(k+k_1)^2-m^2]} \\ &-\int \frac{d^4k}{(2\pi)^4}\frac{(k_2^2+2k_2\cdot k)}{[(k^2-m^2)^2][(k+k_2)^2-m^2]} + \\ &+\int \frac{d^4k}{(2\pi)^4}\frac{(k_1^2+2k_1\cdot k)(k_2^2+2k_2\cdot k)}{[(k^2-m^2)^2][(k+k_1)^2-m^2][(k+k_2)^2-m^2]},\end{aligned} \qquad (64)$$

which is very convenient since it maintains the symmetry in k_1 and k_2. The divergent content of this amplitude is contained in the basic divergent object I_{log}. The remaining integrals are finite and yield

$$\int \frac{d^4k}{(2\pi)^4}\frac{(k_1^2+2k_1\cdot k)}{[(k^2-m^2)^2][(k+k_1)^2-m^2]} = \left(\frac{i}{(4\pi)^2}\right)[(-)Z_0(m^2,m^2,k_1^2;m^2)] \qquad (65)$$

and

$$\int \frac{d^4k}{(2\pi)^4}\frac{(k_1^2+2k_1\cdot k)(k_2^2+2k_2\cdot k)}{[(k^2-m^2)^2][(k+k_1)^2-m^2][(k+k_2)^2-m^2]} =$$
$$\left(\frac{i}{(4\pi)^2}\right)\Big[Z_0(m^2,m^2,k_1^2;m^2)+Z_0(m^2,m^2,k_2^2;m^2)-Z_0(m^2,m^2,(k_1-k_2)^2;m^2)\Big] \qquad (66)$$

where we leave the integration in the last of Feynman parameters through the introduction of the structure functions for one loop integrals defined as [5]

$$Z_k(\lambda_1^2,\lambda_2^2,q^2;\lambda^2) = \int_0^1 dz z^k ln\left(\frac{q^2z(1-z)+(\lambda_1^2-\lambda_2^2)z-\lambda_1^2}{(-\lambda^2)}\right), \qquad (67)$$

which have proven very useful in the systematization of this type of calculations. The logarithmically divergent integral is then

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k+k_1)^2 - m^2][(k+k_2)^2 - m^2]} = I_{log}(m^2) - \left(\frac{i}{(4\pi)^2} \right) Z_0(m^2, m^2, (k_1 - k_2)^2; m^2) \tag{68}$$

and the amplitude T^{SS}

$$\begin{aligned} T^{SS} &= 4 \Bigg\{ [I_{quad}(m^2)] + \frac{4m^2 - (k_1 - k_2)^2}{2} [I_{log}(m^2)] \\ &- \frac{4m^2 - (k_1 - k_2)^2}{2} \left(\frac{i}{(4\pi)^2} \right) Z_0(m^2, m^2, (k_1 - k_2)^2; m^2) \\ &+ \frac{k_{1\alpha} k_{1\beta}}{2} [\triangle_{\alpha\beta}] + \frac{k_{2\rho} k_{2\xi}}{2} [\triangle_{\rho\xi}] \Bigg\}, \end{aligned} \tag{69}$$

where the last two terms are, in principle ambiguous. The finite contributions and one proportional to the basic divergent integral I_{log} are unambiguous since the combination $k_2 - k_1$ is just equal to the external momentum q.

The next amplitude is

$$\begin{aligned} T^{PP} &= \int \frac{d^4k}{(2\pi)^4} Tr \left\{ \gamma_5 \frac{1}{(\not{k} + \not{k}_1) - m} \gamma_5 \frac{1}{(\not{k} + \not{k}_2) - m} \right\} \\ &= 4 \Bigg\{ -[I_{quad}(m^2)] + \frac{(k_1 - k_2)^2}{2} [I_{log}(m^2)] \\ &- \frac{(k_1 - k_2)^2}{2} \left(\frac{i}{(4\pi)^2} \right) Z_0(m^2, m^2, (k_1 - k_2)^2; m^2) \\ &- \frac{k_{1\alpha} k_{1\beta}}{2} [\triangle_{\alpha\beta}] - \frac{k_{2\rho} k_{2\xi}}{2} [\triangle_{\rho\xi}] \Bigg\}. \end{aligned} \tag{70}$$

which also presents potentially ambiguous terms. The amplitude T_μ^{AP}, with the same ingredients, is given by

$$\begin{aligned} T_\mu^{AP} &= \int \frac{d^4k}{(2\pi)^4} Tr \left\{ i\gamma_\mu \gamma_5 \frac{1}{(\not{k} + \not{k}_1) - m} \gamma_5 \frac{1}{(\not{k} + \not{k}_2) - m} \right\} \\ &= -4m(k_1 - k_2)_\mu \left\{ [I_{log}(m^2)] - \left(\frac{i}{(4\pi)^2} \right) Z_0(m^2, m^2, (k_1 - k_2)^2; m^2) \right\}, \end{aligned} \tag{71}$$

that is unambiguous.

The amplitudes

$$
\begin{aligned}
T_{\mu}^{VS} &= \int \frac{d^4k}{(2\pi)^4} Tr \left\{ \gamma_\mu \frac{1}{(\not k + \not k_1) - m} \hat{1} \frac{1}{(\not k + \not k_2) - m} \right\} \\
&= -4m(k_1 + k_2)_\beta [\triangle_{\beta\mu}]
\end{aligned} \tag{72}
$$

and

$$
\begin{aligned}
T_{\mu\nu}^{AV} &= \int \frac{d^4k}{(2\pi)^4} Tr \left\{ i\gamma_\mu \gamma_5 \frac{1}{(\not k + \not k_1) - m} \gamma_\nu \frac{1}{(\not k + \not k_2) - m} \right\} \\
&= -4\varepsilon_{\mu\nu\alpha\beta} \left\{ \frac{k_{1\xi} k_{2\beta}}{2} [\triangle_{\xi\alpha}] + \frac{k_{2\xi} k_{1\alpha}}{2} [\triangle_{\xi\beta}] \right\},
\end{aligned} \tag{73}
$$

can also, in principle, be nonvanishing and ambiguous, i.e., depending on the choice of the labels k_1 and k_2.

Next let us evaluate the axial-axial amplitude

$$
T_{\mu\nu}^{AA} = \int \frac{d^4k}{(2\pi)^4} Tr \left\{ i\gamma_\mu \gamma_5 \frac{1}{[\not k + \not k_1 - m]} i\gamma_\nu \gamma_5 \frac{1}{[\not k + \not k_2 - m]} \right\}, \tag{74}
$$

which, after calculation of the trace allows us to identify the relation

$$
T_{\mu\nu}^{AA} = g_{\mu\nu} [T^{SS}] - T_{\mu\nu} \tag{75}
$$

where we have defined the tensor

$$
T_{\mu\nu} = 4 \int \frac{d^4k}{(2\pi)^4} \frac{[(k+k_1)_\mu (k+k_2)_\nu + (k+k_1)_\nu (k+k_2)_\mu]}{[(k+k_1)^2 - m^2][(k+k_2)^2 - m^2]}. \tag{76}
$$

After some manipulation we get

$$
\begin{aligned}
T_{\mu\nu}^{AA} = & -\frac{4}{3} [(k_1 - k_2)^2 g_{\mu\nu} - (k_1 - k_2)_\mu (k_1 - k_2)_\nu] \times \\
& \times \left\{ [I_{log}(m^2)] - \left(\frac{i}{(4\pi)^2} \right) \left[\frac{1}{3} + \frac{(2m^2 + (k_1 - k_2)^2)}{(k_1 - k_2)^2} Z_0(m^2, m^2, (k_1 - k_2)^2) \right] \right\} \\
& - g_{\mu\nu} 8m^2 \left\{ [I_{log}(m^2)] - \left(\frac{i}{(4\pi)^2} \right) Z_0(m^2, m^2, (k_1 - k_2)^2) \right\} \\
& -4(\nabla_{\mu\nu}) - 4k_{1\alpha} k_{2\beta} \{ \Box_{\alpha\beta\mu\nu} - g_{\nu\beta} \triangle_{\alpha\mu} - g_{\mu\beta} \triangle_{\alpha\nu} - g_{\mu\nu} \triangle_{\alpha\beta} \} \\
& -(k_1 - k_2)_\alpha (k_1 - k_2)_\beta \left\{ \Box_{\alpha\beta\mu\nu} - \frac{g_{\alpha\beta}}{2} \triangle_{\mu\nu} - \frac{g_{\nu\beta}}{2} \triangle_{\alpha\mu} - \frac{g_{\mu\beta}}{2} \triangle_{\alpha\nu} - \frac{g_{\mu\nu}}{2} \triangle_{\alpha\beta} \right\}.
\end{aligned} \tag{77}
$$

The next two point function to be considered is relevant in Electrodynamics, since it is related to the polarization tensor. We therefore expect it to be unambiguous and to have its Ward identities satisfied. It is the vector-vector amplitude given by

$$T_{\mu\nu}^{VV} = \int \frac{d^4k}{(2\pi)^4} Tr\left\{\gamma_\mu \frac{1}{[(\not{k}+\not{k}_1)-m]}\gamma_\nu \frac{1}{[(\not{k}+\not{k}_2)-m]}\right\}, \tag{78}$$

which, after taking the Dirac traces, allows us to identify the relation

$$T_{\mu\nu}^{VV} = g_{\mu\nu}[T^{PP}] + T_{\mu\nu}. \tag{79}$$

Using the corresponding previous results we get

$$\begin{aligned} T_{\mu\nu}^{VV} &= \frac{4}{3}[(k_1-k_2)^2 g_{\mu\nu} - (k_1-k_2)_\mu (k_1-k_2)_\nu] \times \\ &\times \left\{[I_{log}(m^2)] - \left(\frac{i}{(4\pi)^2}\right)\left[\frac{1}{3} + \frac{(2m^2+(k_1-k_2)^2)}{(k_1-k_2)^2} Z_0(m^2, m^2, (k_1-k_2)^2)\right]\right\} \\ &+4(\nabla_{\mu\nu}) + 4k_{1\alpha}k_{2\beta}\left[\Box_{\alpha\beta\mu\nu} - g_{\nu\beta}\triangle_{\alpha\mu} - g_{\mu\beta}\triangle_{\alpha\nu} - g_{\mu\nu}\triangle_{\alpha\beta}\right] \\ &+(k_1-k_2)_\alpha(k_1-k_2)_\beta\left[\Box_{\alpha\beta\mu\nu} - \frac{g_{\alpha\beta}}{2}\triangle_{\mu\nu} - \frac{g_{\nu\beta}}{2}\triangle_{\alpha\mu} - \frac{g_{\mu\beta}}{2}\triangle_{\alpha\nu} - \frac{g_{\mu\nu}}{2}\triangle_{\alpha\beta}\right]. \end{aligned} \tag{80}$$

This calculation completes the evaluation of all nonvanishing two point functions. It is a simple matter to show that the remaining ones T_μ^{VP}, T^{PS} and T_μ^{AS} vanish identically due to the presence os the γ_5 matrix.

4.3 Three Point Functions

We now come to the three point functions. In order not to overload the text we limit ourselves in writing only explicitly terms indicating the presence of ambiguities. A relatively simple case which illustrates what we mean is the following

$$T_\lambda^{VSS} = \int \frac{d^4k}{(2\pi)^4} Tr\left\{\gamma_\lambda \frac{1}{[\not{k}+\not{k}_3-m]}\hat{1}\frac{1}{[\not{k}+\not{k}_2-m]}\hat{1}\frac{1}{[\not{k}+\not{k}_1-m]}\right\} \tag{81}$$

After taking the trace and performing some algebraic reorganization of terms we obtain the expression

$$
\begin{aligned}
T_{\lambda}^{VSS} \quad = \quad & 2\Bigg\{\int \frac{d^4k}{(2\pi)^4}\frac{2k_\lambda}{[(k+k_2)^2-m^2][(k+k_3)^2-m^2]} \\
& +(k_2+k_3)_\lambda\int \frac{d^4k}{(2\pi)^4}\frac{1}{[(k+k_2)^2-m^2][(k+k_3)^2-m^2]} \\
& +(k_3-k_1)_\lambda\int \frac{d^4k}{(2\pi)^4}\frac{1}{[(k+k_1)^2-m^2][(k+k_3)^2-m^2]} \\
& +(k_2-k_1)_\lambda\int \frac{d^4k}{(2\pi)^4}\frac{1}{[(k+k_1)^2-m^2][(k+k_2)^2-m^2]} \\
& +(8m^2-(k_1-k_2)^2-(k_1-k_3)^2+(k_2-k_3)^2)\times \\
& \quad \times\int \frac{d^4k}{(2\pi)^4}\frac{k_\lambda}{[k^2-m^2][(k+k_2-k_1)^2-m^2][(k+k_3-k_1)^2-m^2]} \\
& +\Big[[4m^2-(k_1-k_2)^2](k_3-k_1)_\lambda+[4m^2-(k_1-k_3)^2](k_2-k_1)_\lambda\Big]\times \\
& \quad \times\int \frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-m^2][(k+k_2-k_1)^2-m^2][(k+k_3-k_1)^2-m^2]}\Bigg\}. \qquad (82)
\end{aligned}
$$

In the above expression we note that only the first two terms are potentially ambiguous. The remaining ones are either logarithmically divergent or finite, both with unambiguous coefficients. Of course it is possible to solve all integrals and obtain an analytical expression for T_{λ}^{VSS}. But, for our immediate purposes it is enough to write it in the form

$$T_{\lambda}^{VSS}=-2(k_2+k_3)_\xi[\triangle_{\lambda\xi}]+NAT, \qquad (83)$$

where NAT stands for nonambiguous terms. Having this in mind we simply list the other three point functions in the same manner.

$$T_{\lambda}^{VPP}=2(k_2+k_3)_\xi[\triangle_{\lambda\xi}]+NAT, \qquad (84)$$

with only one Lorentz index we have

$$T_{\mu}^{SAP}=2i(k_1+k_3)_\xi[\triangle_{\mu\xi}]+NAT \qquad (85)$$

On the other hand

$$T_{\lambda\mu\nu}^{AAA}=2i\varepsilon_{\lambda\mu\nu\xi}(k_1+k_2)_\sigma(\triangle_{\xi\sigma})+NAT \qquad (86)$$

and

$$T_{\lambda\mu\nu}^{AVV}=-2i\varepsilon_{\lambda\mu\nu\xi}(k_1+k_2)_\sigma(\triangle_{\xi\sigma})+NAT \qquad (87)$$

which presents ambiguous terms which are identical to those found in $T^{AAA}_{\lambda\mu\nu}$. And finally the more complex ones

$$\begin{aligned} T^{VVV}_{\lambda\mu\nu} &= \Big\{ (k_1+k_2)_\xi \Big[-\frac{2}{3}\Box_{\xi\mu\nu\lambda} - \frac{2}{3} g_{\xi\nu} \triangle_{\mu\lambda} - \frac{2}{3} g_{\xi\mu}\triangle_{\nu\lambda} \\ & -\frac{2}{3} g_{\xi\lambda}\triangle_{\mu\nu} + 2 g_{\lambda\mu}\triangle_{\nu\xi} + 2 g_{\xi\nu}\triangle_{\lambda\mu} \Big] \\ & +(k_1+k_3)_\xi \Big[-\frac{2}{3}\Box_{\xi\mu\nu\lambda} - \frac{2}{3} g_{\xi\nu} \triangle_{\mu\lambda} - \frac{2}{3} g_{\xi\mu}\triangle_{\nu\lambda} \\ & -\frac{2}{3} g_{\xi\lambda}\triangle_{\mu\nu} + 2 g_{\lambda\nu}\triangle_{\mu\xi} + 2 g_{\xi\mu}\triangle_{\lambda\nu} \Big] \\ & +(k_2+k_3)_\xi \Big[-\frac{2}{3}\Box_{\xi\mu\nu\lambda} - \frac{2}{3} g_{\xi\nu} \triangle_{\mu\lambda} - \frac{2}{3} g_{\xi\mu}\triangle_{\nu\lambda} \\ & -\frac{2}{3} g_{\xi\lambda}\triangle_{\mu\nu} + 2 g_{\mu\nu}\triangle_{\lambda\xi} + 2 g_{\xi\lambda}\triangle_{\mu\nu} \Big] + TNA \Big\} . \end{aligned} \tag{88}$$

In order to give a complete account of the three point functions we give the amplitude $T^{VAA}_{\lambda\mu\nu}$, which possesses the same potentially ambiguous term as $T^{VVV}_{\lambda\mu\nu}$, i.e.

$$\left. T^{VAA}_{\lambda\mu\nu} \right|_{PA} = - \left. T^{VVV}_{\lambda\mu\nu} \right|_{PA} . \tag{89}$$

The remaining three point functions are all unambiguous although divergent, therefore are left out of the discussions.

Let us now establish contact between our results and those obtained by Gerstein and Jackiw showing that it is a simple matter to go from our formalism over to theirs.

5 Ambiguities in Gerstein and Jackiw Model

Let us now understand how the results obtained in the previous section include those of ref.[1] as a special case. We show that starting from our expressions we can obtain the results in tables I and II for the ambiguities in the two and three point functions as defined by those authors in section III of ref.[1]. We start with our results for the amplitude T^{VS}_{μ} given by eq.(73) as

$$T^{VS}_{\mu} = -4m(k_1+k_2)_\beta [\triangle_{\beta\mu}]. \tag{90}$$

In the present notation the ambiguity has been introduced via the arbitrary momenta k_1 and k_2, by calculating the amplitudes with internal momenta in the loop: $k+k_1$ and $k+k_2$ in the two propagators in question, as defined in eq.(6). In ref.[1] the arbitrariness is represented by the momentum s, such that the propagators have momenta $k+s$ and $k+s+p$ which establish the equivalence relation

$$\begin{cases} k_1 = s \\ k_2 = s+p \end{cases}$$

In this way the ambiguous part of the amplitude can be cast in the form

$$T_{\mu}^{VS} = -4m(2s)_{\beta}\left\{\int_{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{4k_{\mu}k_{\beta}}{(k^2-m^2)^3} - \int_{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{g_{\mu\beta}}{(k^2-m^2)^2}\right\} \tag{91}$$

where we have used definitions in eq.(58) $\triangle_{\beta\mu}$. Taking now the limit in a symmetric way in the first integral, i.e., using the relation

$$\lim_{k\to\infty} k_{\mu}k_{\beta} = \frac{1}{4}k^2 g_{\mu\beta} \tag{92}$$

and taking $k^2 \to k^2 - m^2 + m^2$ in this integral we get

$$T_{\mu}^{VS} = -4m(2s)_{\beta}\left\{\int\frac{d^4k}{(2\pi)^4}\frac{m^2 g_{\mu\beta}}{(k^2-m^2)^3}\right\}. \tag{93}$$

Using the result

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)^3} = \frac{i}{2(4\pi)^2(-m^2)} \tag{94}$$

we finally get

$$T_{\mu}^{VS} = \frac{4i\pi^2}{(2\pi)^4}ms_{\mu}, \tag{95}$$

as in ref.[1]. In an analogous way we get for the ambiguous contribution of T^{PP} and T^{SS}

$$T^{PP}|_{AP} = T^{SS}|_{AP} = \frac{-i\pi^2}{(2\pi)^4}s\cdot(s+p), \tag{96}$$

and for $T_{\mu\nu}^{AV}$

$$T_{\mu\nu}^{AV} = \frac{-2\pi^2\varepsilon_{\mu\alpha\nu\beta}s_{\alpha}p_{\beta}}{(2\pi)^4}. \tag{97}$$

The functions $T_{\mu\nu}^{VV}$ and $T_{\mu\nu}^{AA}$ have the same ambiguous part given by

$$T_{\mu\nu}^{VV}|_{AP} = T_{\mu\nu}^{AA}|_{AP} = 4k_{1\alpha}k_{2\beta}\left[\Box_{\alpha\beta\mu\nu} - g_{\nu\beta}\triangle_{\alpha\mu} - g_{\mu\beta}\triangle_{\alpha\nu} - g_{\mu\nu}\triangle_{\alpha\beta}\right], \tag{98}$$

where $\Box_{\alpha\beta\mu\nu}$ is defined in eq.(56). In this case it is necessary to use the relation

$$\lim_{k\to\infty}\frac{k_{\alpha}k_{\beta}k_{\mu}k_{\nu}}{k^4} = \frac{1}{24}\left(g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu}\right) \tag{99}$$

and also the following relations

$$\int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - m^2)^4} = \frac{i}{(4\pi)^2} \frac{1/2 g_{\mu\nu}}{3!(-m^2)} \tag{100}$$

and

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^2 - m^2)^4} = \frac{i}{(4\pi)^2} \frac{1}{3(-m^2)} \tag{101}$$

in order to obtain (after some algebra)

$$T^{VV}_{\mu\nu}|_{AP} = T^{AA}_{\mu\nu}|_{AP} = \frac{2i\pi^2}{3(2\pi)^4} \left\{ s_\mu (s+p)_\nu + g_{\mu\nu} s \cdot (s+p) + s_\nu (s+p)_\mu \right\}. \tag{102}$$

In the case of the three point functions, in order to obtain the results of ref.[1] it is necessary to take the SU(3) indices into account in the calculation of the amplitudes, since they play an important role in the construction of currents. The Gell-Mann matrices introduce symmetry factors such as relative global signs between the direct and crossed channel, which are not present in our nonabelian model. However this does not introduce any inconsistency in the two formulations, as we will see.

The arbitrariness in the label for the internal momenta of the loop is introduced by taking

$$s = ap + bp' \tag{103}$$

where a and b are real numbers. The ambiguity is than found by means of calculating the difference between two representations for the three point functions: with and without the arbitrary label s defined above. In this sense we see that it can be interpreted as a surface term, which would disappear were shifts in the integration variable are allowed.

Following this prescription, the ambiguous part of T^{SAP}_μ previously found in the form

$$T^{S\to AP}_\mu|_{AP} = 2i(k_1 + k_3)_\xi \triangle_{\mu\xi} \tag{104}$$

for the direct channel (without SU(3) factors) may be written as

$$T^{S\to AP}_\mu|_{AP} = \frac{2\pi^2}{(2\pi)^4}(ap + bp')_\mu \tag{105}$$

which, after the inclusion of the crossed channel and the pertinent factors yield

$$T^{S\to AP}_\mu|_{AP} = \frac{2\pi^2}{(2\pi)^4}(a-b)(p-p')_\mu. \tag{106}$$

In a completely analogous way we get

$$T_{\mu}^{V\to SS}|_{AP} = T_{\mu}^{V\to PP}|_{AP} = \frac{-i2\pi^2}{(2\pi)^4}(a-b)(p-p')_{\mu}, \tag{107}$$

$$T_{\mu\alpha\beta}^{A\to AA}|_{AP} = T_{\mu\alpha\beta}^{A\to VV}|_{AP} = \frac{-2\pi^2}{(2\pi)^4}(b-a)\varepsilon_{\mu\alpha\beta\lambda}(p-p')_{\lambda}, \tag{108}$$

and finally

$$T_{\mu\alpha\beta}^{V\to VV}|_{AP} = T_{\mu\alpha\beta}^{V\to AA}|_{AP} = \frac{-2i\pi^2}{3(2\pi)^4}(b-a)\left\{g_{\mu\alpha}(p-p')_{\beta} + g_{\mu\beta}(p-p')_{\alpha} + g_{\alpha\beta}(p-q)_{\mu}\right\} \tag{109}$$

6 Ward Identities

We next proceed to the verification of the identities given by eqs.(12)-(46) using the free fermion model. Those relations were obtained formally by contracting the Lorentz index of the vertex with the respective external momentum. There are two ways to arrive at the final result: firstly, one calculates the Green's functions with lower number of points as those in the function one is investigating. Secondly, one can explicitly evaluate the Green's function in question and afterwards contracting the results with the appropriate external momentum. We will analyze both possibilities for the two point functions and only the first one for the three point functions. For this purpose we perform, as before, only algebraic operations in order to express the four-divergencies in the form of Ward identities.

6.1 One Point Functions

The first Ward identity to which we refer in section 2 is T_{μ}^{V}. It is a simple matter to check that in order to obtain eq.(12) we should require (from eq.(55)) the following conditions to be satisfied

$$\begin{cases} \Box_{\alpha\beta\mu\nu} = 0 \\ \nabla_{\mu\nu} = 0 \\ \triangle_{\mu\nu} = 0. \end{cases} \tag{110}$$

Since all terms of T_{μ}^{V} in eq.(55) are ambiguous, requiring relations eqs.(110) rendering the amplitude as unambiguous and preserving symmetry. As will be shown in what follows it is necessary and sufficient to "save" all other amplitudes from ambiguities and symmetry violations. There is of course the immediate question for the possibility of an existing regularization scheme which turns eqs.(110) possible. The answer is positive and some possibilities are given in ref.[12].

6.2 Two Point Functions

We now study the Ward identities for the two point functions. Let us, initially, consider the contraction of T_μ^{VS} with the external momentum $(k_1 - k_2)_\mu$

$$(k_1 - k_2)_\mu T_\mu^{VS} = \int \frac{d^4k}{(2\pi)^4} Tr \left\{ \hat{1} \frac{1}{[\not{k} + \not{k}_1 - m]} (\not{k}_1 - \not{k}_2) \frac{1}{[\not{k} + \not{k}_2 - m]} \right\}. \tag{111}$$

Before calculating the trace we use the following identity

$$(\not{k}_1 - \not{k}_2) = [\not{k} + \not{k}_1 - m] - [\not{k} + \not{k}_2 - m], \tag{112}$$

so that

$$(k_1 - k_2)_\mu T_\mu^{VS} = \int \frac{d^4k}{(2\pi)^4} Tr \left\{ \hat{1} \frac{1}{[\not{k} + \not{k}_2 - m]} \right\} - \int \frac{d^4k}{(2\pi)^4} Tr \left\{ \hat{1} \frac{1}{[\not{k} + \not{k}_1 - m]} \right\}. \tag{113}$$

Comparing now with eq.(59) we identify two scalar one point Green's functions. It is clear that the vector current will only be conserved if there is no dependence on k_1 and k_2 in the one point functions. However, using the result in eq.(61) for T^S we get

$$(k_1 - k_2)_\mu T_\mu^{VS} = 4m(k_{2\alpha} k_{2\beta} - k_{1\alpha} k_{1\beta})[\triangle_{\alpha\beta}]. \tag{114}$$

Apparently, in what concerns possible choices for the k_1 and k_2 values there is no way the relation $(k_1 - k_2)_\mu T_\mu^{VS} = 0$ to be satisfied.

We can, on the other hand, also check the validity of the identity by the contraction of $(k_1 - k_2)_\mu$ with the expression obtained for T_μ^{VS}, eq.(73):

$$(k_1 - k_2)_\mu T_\mu^{VS} = -4m(k_1 - k_2)_\mu (k_1 + k_2)_\beta [\triangle_{\mu\beta}]. \tag{115}$$

In any case the identity can only be satisfied provided $\triangle_{\mu\beta} = 0$, which renders T^S unambiguous and T_μ^{VS} zero, as one can check from eq.(61) e eq.(73). Let us now study the case of axial-pseudoscalar amplitude T_μ^{AP}, defined in eq.(71). Firstly we do the following

$$(k_1 - k_2)_\mu T_\mu^{AP} = \int \frac{d^4k}{(2\pi)^4} Tr \left\{ \gamma_5 \frac{1}{[\not{k} + \not{k}_2 - m]} i(\not{k}_1 - \not{k}_2)\gamma_5 \frac{1}{[\not{k} + \not{k}_1 - m]} \right\}, \tag{116}$$

and introduce the identity

$$(\not{k}_1 - \not{k}_2)\gamma_5 = [\not{k} + \not{k}_1 - m]\gamma_5 + \gamma_5[\not{k} + \not{k}_2 - m] + 2m\gamma_5 \tag{117}$$

in the interior of the trace, which leads us to

$$\begin{aligned}(k_1 - k_2)_\mu T^{AP}_\mu &= -2miT^{PP} - i\int \frac{d^4k}{(2\pi)^4} Tr\left\{\hat{1}\frac{1}{[\not{k} + \not{k}_1 - m]}\right\} \\ &\quad -i\int \frac{d^4k}{(2\pi)^4} Tr\left\{\hat{1}\frac{1}{[\not{k} + \not{k}_2 - m]}\right\}. \end{aligned} \tag{118}$$

Again we see that the verification of the Ward identity will crucially depend on values assumed by the scalar one point function. Now it is required that the sum of both do not depend on k_1 and k_2. Substituting the results previously obtained for the one point function we obtain

$$(k_1 - k_2)_\mu T^{AP}_\mu = -2miT^{PP} - 2mi\left\{4I_{quad}(m^2) + 2[k_{1\alpha}k_{1\beta} + k_{2\alpha}k_{2\beta}]\triangle_{\alpha\beta}\right\}. \tag{119}$$

Had we, on the other hand, taken eq.(71) and contracting it with $(k_1 - k_2)_\mu$, we would have gotten

$$(k_1 - k_2)_\mu T^{AP}_\mu = -4mi(k_1 - k_2)^2\left\{[I_{log}(m^2)] - \left(\frac{i}{(4\pi)^2}\right) Z_0(m^2, m^2, (k_1 - k_2)^2; m^2)\right\}, \tag{120}$$

from which we could get

$$\begin{aligned}(k_1 - k_2)_\mu T^{AP}_\mu &= -2mi\left\{-4[I_{quad}(m^2)] + 2(k_1 - k_2)^2[I_{log}(m^2)]\right. \\ &\quad - \left(\frac{i}{(4\pi)^2}\right)(k_1 - k_2)^2 2Z_0(m^2, m^2, (k_1 - k_2)^2; m^2) \\ &\quad \left. -2(k_{1\alpha}k_{1\beta} + k_{2\alpha}k_{2\beta})\triangle_{\alpha\beta}\right\} \\ &\quad -2mi\left\{4[I_{quad}(m^2)] + 2(k_{1\alpha}k_{1\beta} + k_{2\alpha}k_{2\beta})\triangle_{\alpha\beta}\right\} \end{aligned} \tag{121}$$

This results agrees with eq.(101) provided we identify, in the first term, in curly brackets, the expression for T^{PP}, as we can see from eq.(70). The Ward identity in eq.(16), apparently, will be violated by ambiguous contributions.

Let us next take the amplitude $T^{AV}_{\mu\nu}$ which has two Lorentz indices and has two Ward identities connected to it, one for the vector index and other for the axial one. For the vector current we have

$$\begin{aligned}(k_1 - k_2)_\nu T^{AV}_{\mu\nu} &= \int \frac{d^4k}{(2\pi)^4} Tr\left\{i\gamma_\mu\gamma_5\frac{1}{[\not{k} + \not{k}_2 - m]}\right\} \\ &\quad -\int \frac{d^4k}{(2\pi)^4} Tr\left\{i\gamma_\mu\gamma_5\frac{1}{[\not{k} + \not{k}_1 - m]}\right\}, \end{aligned} \tag{122}$$

where we immediately identify the one point axial functions which vanish identically and therefore $(k_1 - k_2)_\nu T^{AV}_{\mu\nu} = 0$. Doing the same for the axial current we have

$$\begin{aligned}(k_1 - k_2)_\mu T^{AV}_{\mu\nu} &= -2mi[T^{PV}_\nu] - \int \frac{d^4k}{(2\pi)^4} Tr\left\{ i\gamma_\nu\gamma_5 \frac{1}{[\not{k} + \not{k}_2 - m]} \right\} \\ &\quad - \int \frac{d^4k}{(2\pi)^4} Tr\left\{ i\gamma_\nu\gamma_5 \frac{1}{[\not{k} + \not{k}_1 - m]} \right\}, \qquad (123)\end{aligned}$$

so that

$$(k_1 - k_2)_\mu T^{AV}_{\mu\nu} = -2mi[T^{PV}_\nu]. \qquad (124)$$

yields the expected result for the Ward identity in eq.(17). However the amplitude T^{PV}_ν is identically zero given Dirac trace properties, which immediatly implies with $T^{AV}_{\mu\nu} = 0$. Checking now with eq.(74) we see that the result obtained for the identity using explicit expression for the amplitude and then effecting the contraction forces that $\triangle_{\mu\nu} = 0$. Only in this case consistency is restored. If we take the four-divergence directly from eq.(74) it is easy see that $(k_1 - k_2)_\nu T^{AV}_{\mu\nu} = 0$ and $(k_1 - k_2)_\mu T^{AV}_{\mu\nu} = 0$.

The case of the amplitude $T^{VV}_{\mu\nu}$ is analogous. If we first perform the contraction

$$\begin{aligned}(k_1 - k_2)_\mu T^{VV}_{\mu\nu} &= \int \frac{d^4k}{(2\pi)^4} Tr\left\{ \gamma_\nu \frac{1}{[\not{k} + \not{k}_2 - m]} \right\} \\ &\quad - \int \frac{d^4k}{(2\pi)^4} Tr\left\{ \gamma_\nu \frac{1}{[\not{k} + \not{k}_1 - m]} \right\}, \qquad (125)\end{aligned}$$

we again see that the result will depend on the one point function. Substituting now for their expressions we get

$$\begin{aligned}(k_1 - k_2)_\nu T^{VV}_{\mu\nu} &= 4\left\{ (-)k_{2\alpha}\nabla_{\alpha\mu} - \frac{k_{2\alpha}k_{2\beta}k_{2\rho}}{3}\Box_{\alpha\beta\rho\mu} \right. \\ &\quad \left. - \frac{k_{2\mu}k_{2\alpha}k_{2\beta}}{3}\triangle_{\alpha\beta} + \frac{1}{3}k_2^2 k_{2\rho}\triangle_{\rho\mu} + k_{2\mu}k_{2\alpha}k_{2\beta}\triangle_{\alpha\beta} \right\} \\ &\quad -4\left\{ (-)k_{1\alpha}\nabla_{\alpha\mu} - \frac{k_{1\alpha}k_{1\beta}k_{1\rho}}{3}\Box_{\alpha\beta\rho\mu} \right. \\ &\quad \left. - \frac{k_{1\mu}k_{1\alpha}k_{1\beta}}{3}\triangle_{\alpha\beta} + \frac{1}{3}k_1^2 k_{1\rho}\triangle_{\rho\mu} + k_{1\mu}k_{1\alpha}k1_\beta\triangle_{\alpha\beta} \right\}. \qquad (126)\end{aligned}$$

The conservation of the vector current demands, therefore that the r.h.s. be identically null, which cannot be obtained by any choice of k_1 and k_2. It is easy to check that the same result would have been obtained if we had taken the four divergence directly on the final result for $T^{VV}_{\mu\nu}$, eq.(82), since the unambiguous term of the latter is Gauge invariant. This shows that in order to satisfy the Ward identities relative to the $T^{VV}_{\mu\nu}$ amplitude we need to require the same conditions as for the one point function, eq.(92).

Last we turn to $T^{AA}_{\mu\nu}$. Associated to this Green's function we have two axial currents. This allows us to relate them to the amplitudes T^{AP}_{μ} and T^{PP} by successive contractions with the external momentum. Thus, upon contracting with $(k_1-k_2)_\mu$ we get

$$\begin{aligned}(k_1-k_2)_\mu T^{AA}_{\mu\nu} &= -2mi[T^{PA}_{\nu}] + \int \frac{d^4k}{(2\pi)^4} Tr\left\{\gamma_\nu \frac{1}{[\not k + \not k_1 - m]}\right\} \\ &\quad - \int \frac{d^4k}{(2\pi)^4} Tr\left\{\gamma_\nu \frac{1}{[\not k + \not k_2 - m]}\right\}, \end{aligned} \tag{127}$$

Where the two last terms are again vector one point functions and we get

$$\begin{aligned}(k_1-k_2)_\mu T^{AA}_{\mu\nu} &= 4\left\{-\left[-k_{1\beta}\nabla_{\beta\nu} - k_{1\alpha}k_{1\beta}k_{1\mu}\Box_{\alpha\beta\nu\mu}\right.\right. \\ &\quad \left. -\frac{k_{1\nu}k_{1\alpha}k_{1\beta}}{3}\triangle_{\alpha\beta} + \frac{1}{3}k_1^2 k_{1\rho}\triangle_{\rho\nu} + k_{1\nu}k_{1\alpha}k1_{\beta}\triangle_{\alpha\beta}\right] \\ &\quad + \left[-k_{2\alpha}\triangle_{\alpha\nu} - k_{2\alpha}k_{2\beta}k_{2\rho}\Box_{\alpha\beta\rho\nu}\right. \\ &\quad \left.\left. -\frac{k_{2\nu}k_{2\alpha}k_{2\beta}}{3}\triangle_{\alpha\beta} + \frac{1}{3}k_2^2 k_{2\rho}\triangle_{\rho\nu}\right]\right\} \\ &\quad -2miT^{PA}_{\nu}. \end{aligned} \tag{128}$$

Therefore it is verified that the conditions under which the Ward identities are satisfied are the same as the previous ones. In order to obtain the identity from eq.(20) it is enough to contract, once more, with the index ν.

6.3 Three Point Functions

We next turn to the question of the verification of Ward identities related to the three point functions. We will use only the first way, i.e., by relating the contracted functions with two-point functions, following Gerstein and Jackiw [1].

Let us consider the identity for the amplitude T^{VSS}_{λ}

$$(k_3-k_2)_\lambda T^{VSS}_{\lambda} = \int \frac{d^4k}{(2\pi)^4} Tr\left\{\hat{1}\frac{1}{[\not k + \not k_1 - m]}\hat{1}\frac{1}{[\not k + \not k_2 - m]}(\not k_3 - \not k_2)\frac{1}{[\not k + \not k_3 - m]}\right\} \tag{129}$$

which can be written in the form

$$
\begin{aligned}
(k_3-k_2)_\lambda T_\lambda^{VSS} &= \int \frac{d^4k}{(2\pi)^4} Tr\left\{ \hat{1} \frac{1}{[\not k + \not k_1 - m]} \hat{1} \frac{1}{[\not k + \not k_2 - m]} \right\} \\
&\quad - \int \frac{d^4k}{(2\pi)^4} Tr\left\{ \hat{1} \frac{1}{[\not k + \not k_1 - m]} \hat{1} \frac{1}{[\not k + \not k_3 - m]} \right\}, \qquad (130)
\end{aligned}
$$

where one can identify the scalar two-point function i.e.,

$$
(k_3-k_2)_\lambda T_\lambda^{VSS}(k_1,m;k_2,m;k_3,m) = T^{SS}(k_1,m;k_2,m) - T^{SS}(k_1,m;k_3,m). \qquad (131)
$$

Looking at eq.(69) we verify that the amplitude T^{SS} possesses unambiguous terms, i.e., terms which depends on differences $(k_1-k_2)^2$ and $(k_1-k_3)^2$ which are the external momenta p'^2 and p^2 respectively. Explicitly the difference in eq.(13) can be put into the form

$$
\begin{aligned}
&(k_3-k_2)_\lambda T_\lambda^{VSS}(k_1,m;k_2,m;k_3,m) = \\
&\quad 2\left\{ (4m^2-p'^2)\left[I_{log}(m^2) - \left(\frac{i}{(4\pi)^2}\right) Z_0(m^2,m^2,p'^2;m^2) \right] \right. \\
&\quad \left. - (4m^2-p^2)\left[I_{log}(m^2) - \left(\frac{i}{(4\pi)^2}\right) Z_0(m^2,m^2,p^2;m^2) \right] \right\} \\
&\quad +2(k_{2\alpha}k_{2\beta} - k_{3\alpha}k_{3\beta})\triangle_{\alpha\beta}. \qquad (132)
\end{aligned}
$$

Now we need to include the crossed channel. Redefining adequately the momenta of the external lines and operating in a analogous way we finally arrive at

$$
\begin{aligned}
&(l_3-l_2)_\lambda T_\lambda^{VSS}(l_1,m;l_2,m;l_3,m) = \\
&\quad 2\left\{ (4m^2-p^2)\left[I_{log}(m^2) - \left(\frac{i}{(4\pi)^2}\right) Z_0(m^2,m^2,p^2;m^2) \right] \right. \\
&\quad \left. - (4m^2-p'^2)\left[I_{log}(m^2) - \left(\frac{i}{(4\pi)^2}\right) Z_0(m^2,m^2,p'^2;m^2) \right] \right\} \\
&\quad +2(l_{2\alpha}l_{2\beta} - l_{3\alpha}l_{3\beta})\triangle_{\alpha\beta}. \qquad (133)
\end{aligned}
$$

The sum of the contributions of the two channels yield the following expression for the searched Ward identity

$$
q_\lambda T_\lambda^{V\rightarrow SS} = 2(k_{2\alpha}k_{2\beta} - k_{3\alpha}k_{3\beta})\triangle_{\alpha\beta} + 2(l_{2\alpha}l_{2\beta} - l_{3\alpha}l_{3\beta})\triangle_{\alpha\beta}. \qquad (134)
$$

This result shows that the conservation of the vector current depends in this case on the difference $\triangle_{\alpha\beta}$. This condition has already been found previously.

For the amplitude $T_{\lambda}^{V\to PP}$ we find

$$q_{\lambda}T_{\lambda}^{V\to PP}=2(k_{3\alpha}k_{3\beta}-k_{2\alpha}k_{2\beta})\triangle_{\alpha\beta}+2(l_{3\alpha}l_{3\beta}-l_{2\alpha}l_{2\beta})\triangle_{\alpha\beta}. \tag{135}$$

Still with one Lorentz index we have the process $A\to SP$ for which we write

$$(k_3-k_2)_{\lambda}T_{\lambda}^{ASP}=\int\frac{d^4k}{(2\pi)^4}Tr\left\{\hat{1}\frac{1}{[\not k+\not k_1-m]}\gamma_5\frac{1}{[\not k+\not k_2-m]}i(\not k_3-\not k_2)\frac{1}{[\not k+\not k_3-m]}\right\}. \tag{136}$$

Again, making use of the identity

$$(\not k_2-\not k_3)\gamma_5=\gamma_5[\not k+\not k_3-m]+[\not k+\not k_2-m]\gamma_5+2m\gamma_5, \tag{137}$$

with which we can verify on the r.h.s. the two point functions T^{PP} and T^{SS}, besides T^{PSP}. Substituting the results for them with inclusion of the crossed channel, we have

$$q_{\lambda}T_{\lambda}^{A\to SP}=-2mi[T^{P\to SP}]-2(k_{3\alpha}k_{3\beta}-k_{2\alpha}k_{2\beta})\triangle_{\alpha\beta}-2(l_{3\alpha}l_{3\beta}-l_{2\alpha}l_{2\beta})\triangle_{\alpha\beta}, \tag{138}$$

where the identity violating term is again associated with ambiguities.

The cases with two Lorentz indices are now in order. Making use of the two point functions T_{μ}^{VS}, for $T_{\mu\nu}^{SVV}$ we get for the two associated Ward identities

$$\begin{aligned} p_{\mu}T_{\mu\nu}^{S\to VV} &= 4m(k_3-k_1)_{\alpha}\triangle_{\alpha\nu}+4m(l_3-l_1)_{\alpha}\triangle_{\alpha\nu} \\ &= 4m(p+p')_{\alpha}\triangle_{\alpha\nu}, \end{aligned} \tag{139}$$

and

$$\begin{aligned} p'_{\nu}T_{\mu\nu}^{S\to VV} &= 4m(k_1-k_2)_{\alpha}\triangle_{\alpha\mu}+4m(l_1-l_2)_{\alpha}\triangle_{\alpha\mu} \\ &= 4m(p+p')_{\alpha}\triangle_{\alpha\mu}. \end{aligned} \tag{140}$$

In this results it is important to note an unambiguous character for the violation term. This means that the identity depends on the value of $\triangle_{\alpha\mu}$ and this is <u>not</u> associated with ambiguities here.

For the $S\to AA$ process, which has two related axial currents we get

$$p_{\mu}T_{\mu\nu}^{S\to AA}=2mi[T_{\nu}^{S\to PA}]-4m(k_1+k_3)_{\alpha}\triangle_{\alpha\nu}-4m(l_1+l_3)_{\alpha}\triangle_{\alpha\nu} \tag{141}$$

and

$$p'_{\nu}T_{\mu\nu}^{S\to AA}=2mi[T_{\mu}^{S\to AP}]-4m(k_1+k_2)_{\alpha}\triangle_{\alpha\mu}-4m(l_1+l_2)_{\alpha}\triangle_{\alpha\mu}, \tag{142}$$

which are violated by $\triangle_{\mu\nu}$ with ambiguous coefficients.

Let us now look at the process $V \rightarrow AP$. In the case of the vector current, the identity is expressed in terms of the two point functions T_{μ}^{AP} which are unambiguous. Upon inclusion of the crossed channel we promptly obtain

$$q_{\lambda} T_{\lambda\mu}^{V\rightarrow AP} = 0. \tag{143}$$

On the other hand, the identity related to the axial current yields

$$p_{\mu} T_{\lambda\mu}^{V\rightarrow AP} = 2mi[T_{\lambda}^{V\rightarrow PP}] - 4m(k_2+k_3)_{\alpha}\triangle_{\alpha\lambda} - 4m(l_2+l_3)_{\alpha}\triangle_{\alpha\lambda}. \tag{144}$$

The two identities related to $T_{\mu\nu}^{PVV}$ are satisfied without restrictions since the two point functions vanish identically. Thus

$$(k_3-k_1)_{\mu} T_{\mu\nu}^{PVV} = 0, \tag{145}$$

and

$$(k_1-k_2)_{\nu} T_{\mu\nu}^{PVV} = 0. \tag{146}$$

Also for the processes $V \rightarrow AS$ and $P \rightarrow AA$ both Ward identities are satisfied by the same reason. Thus

$$p_{\mu} T_{\mu\nu}^{S\rightarrow AV} = 2mi T_{\nu}^{S\rightarrow PV} \tag{147}$$

and

$$p'_{\nu} T_{\mu\nu}^{S\rightarrow AV} = 0 \tag{148}$$

and also

$$\begin{cases} p_{\mu} T_{\mu\nu}^{P\rightarrow AA} = 2mi T_{\nu}^{P\rightarrow PA} \\ p'_{\nu} T_{\mu\nu}^{P\rightarrow AA} = 2mi T_{\mu}^{P\rightarrow AP} \end{cases} \tag{149}$$

We are now left with the amplitudes with three Lorentz indices. In the case of $T_{\lambda\mu\nu}^{VVV}$ the identities are expressed in terms of the two point functions $T_{\mu\nu}^{VV}$ and we obtain

$$q_{\lambda} T_{\lambda\mu\nu}^{V\rightarrow VV} = 4\left[k_{1\alpha}(k_2-k_3)_{\beta} + l_{1\alpha}(l_2-l_3)_{\beta}\right]\left[\square_{\alpha\beta\mu\nu} - g_{\nu\beta}\triangle_{\alpha\mu} - g_{\mu\beta}\triangle_{\alpha\nu} - g_{\mu\nu}\triangle_{\alpha\beta}\right], \tag{150}$$

$$p_{\mu} T_{\lambda\mu\nu}^{V\rightarrow VV} = 4\left[k_{2\alpha}(k_1-k_3)_{\beta} + l_{3\alpha}(l_2-l_1)_{\beta}\right]\left[\square_{\alpha\beta\nu\lambda} - g_{\beta\lambda}\triangle_{\alpha\nu} - g_{\nu\beta}\triangle_{\alpha\lambda} - g_{\nu\lambda}\triangle_{\alpha\beta}\right] \tag{151}$$

and

$$p'_\nu T^{V\to VV}_{\lambda\mu\nu} = 4\left[k_{3\alpha}(k_2-k_1)_\beta + l_{2\alpha}(l_1-l_3)_\beta\right]\left[\Box_{\alpha\beta\mu\lambda} - g_{\mu\beta}\triangle_{\alpha\lambda} - g_{\lambda\beta}\triangle_{\alpha\mu} - g_{\alpha\beta}\triangle_{\lambda\mu}\right] \tag{152}$$

All three are totally ambiguous and expressed in terms of $\triangle_{\lambda\mu}$ and $\Box_{\alpha\beta\nu\lambda}$.

In the case of the $T^{VAA}_{\lambda\mu\nu}$ amplitude we have analogous results: for the vector current we find

$$q_\lambda T^{V\to AA}_{\lambda\mu\nu} = -4\left[k_{1\alpha}(k_2-k_3)_\beta + l_{1\alpha}(l_2-l_3)_\beta\right]\left[\Box_{\alpha\beta\mu\nu} - g_{\nu\beta}\triangle_{\alpha\mu} - g_{\mu\beta}\triangle_{\alpha\nu} - g_{\mu\nu}\triangle_{\alpha\beta}\right] \tag{153}$$

and for the axial currents

$$\begin{aligned} p_\mu T^{V\to AA}_{\lambda\mu\nu} &= 2mi[T^{V\to PA}_{\lambda\nu}] \\ &\quad +4i\left[k_{2\beta}(k_1-k_3)_\alpha + l_{3\alpha}(l_2-l_1)_\beta\right]\left[\Box_{\alpha\beta\lambda\nu} - g_{\nu\beta}\triangle_{\lambda\alpha} - g_{\lambda\beta}\triangle_{\nu\alpha} - g_{\lambda\nu}\triangle_{\alpha\beta}\right] \end{aligned} \tag{154}$$

and

$$\begin{aligned} p'_\nu T^{V\to AA}_{\lambda\mu\nu} &= 2mi[T^{V\to AP}_{\lambda\mu}] \\ &\quad +4i\left[k_{3\alpha}(k_2-k_1)_\beta + l_{2\beta}(l_1-l_3)_\alpha\right]\left[\Box_{\alpha\beta\lambda\mu} - g_{\mu\beta}\triangle_{\lambda\alpha} - g_{\lambda\beta}\triangle_{\mu\alpha} - g_{\lambda\mu}\triangle_{\alpha\beta}\right]. \end{aligned} \tag{155}$$

Let us now consider the amplitude $T^{AVV}_{\lambda\mu\nu}$. We start by treating the axial current associated to the index λ. Contracting with the corresponding external momentum, after substituting the two point functions $T^{AV}_{\mu\nu}$ and adding the crossed channel we get

$$\begin{aligned} q_\lambda T^{A\to VV}_{\lambda\mu\nu} &= -2mi[T^{P\to VV}_{\mu\nu}] \\ &\quad -4\varepsilon_{\mu\alpha\nu\beta}\left[(k_{1\xi}k_{2\beta} + k_{3\xi}k_{1\beta})\triangle_{\xi\alpha} + (k_{2\xi}k_{1\alpha} + k_{1\xi}k_{3\alpha})\triangle_{\xi\beta}\right] \\ &\quad +4\varepsilon_{\mu\alpha\nu\beta}\left[(l_{3\xi}l_{1\beta} + l_{1\xi}l_{2\beta})\triangle_{\xi\alpha} + (l_{1\xi}l_{3\alpha} + l_{2\xi}l_{1\alpha})\triangle_{\xi\beta}\right]. \end{aligned} \tag{156}$$

In the same way for the vector currents we arrive at

$$\begin{aligned} p_\mu T^{A\to VV}_{\lambda\mu\nu} &= -4\varepsilon_{\lambda\alpha\nu\beta}\left[k_{2\beta}(k_1-k_3)_\xi\triangle_{\xi\alpha} + k_{2\xi}(k_1-k_3)_\alpha\triangle_{\xi\beta}\right] \\ &\quad -4\varepsilon_{\lambda\alpha\nu\beta}\left[l_{3\xi}(l_1-l_2)_\beta\triangle_{\xi\alpha} + l_{3\alpha}(l_1-l_2)_\xi\triangle_{\xi\beta}\right] \end{aligned} \tag{157}$$

and

$$p'_\nu T^{A\to VV}_{\lambda\mu\nu} = -2\varepsilon_{\lambda\alpha\mu\beta}\left[k_{3\xi}(k_2-k_1)_\beta\triangle_{\xi\beta}+k_{3\alpha}(k_2-k_1)_\xi\triangle_{\xi\beta}\right] \\ -2\varepsilon_{\lambda\alpha\mu\beta}\left[l_{2\beta}(l_1-l_3)_\xi\triangle_{\xi\alpha}+l_{2\xi}(l_1-l_3)_\alpha\triangle_{\xi\beta}\right]. \tag{158}$$

We finally repeat the procedure for the three Ward identities associated to $T^{AAA}_{\lambda\mu\nu}$. In this case, the contraction with the external momenta give rise to the two point functions $T^{AV}_{\mu\nu}$ which, after some manipulation yields

$$q_\lambda T^{A\to AA}_{\lambda\mu\nu} = -2miT^{P\to AA}_{\mu\nu} \\ -4\varepsilon_{\mu\alpha\nu\beta}\left[(k_{1\xi}k_{3\beta}+k_{2\xi}k_{1\beta})\triangle_{\xi\alpha}+(k_{3\xi}k_{1\alpha}+k_{1\xi}k_{2\alpha})\triangle_{\xi\beta}\right] \\ +4\varepsilon_{\mu\alpha\nu\beta}\left[(l_{2\xi}l_{1\beta}+l_{1\xi}l_{3\beta})\triangle_{\xi\alpha}+(l_{1\xi}l_{2\alpha}+l_{3\xi}l_{1\alpha})\triangle_{\xi\beta}\right] \tag{159}$$

and in a completely analogous fashion

$$p_\mu T^{A\to AA}_{\lambda\mu\nu} = 2miT^{A\to PA}_{\lambda\nu} \\ -4\varepsilon_{\mu\alpha\nu\beta}\left[(k_{2\xi}k_{1\beta}+k_{3\xi}k_{1\beta})\triangle_{\xi\alpha}+(k_{1\xi}k_{2\alpha}+k_{2\xi}k_{3\alpha})\triangle_{\xi\beta}\right] \\ +4\varepsilon_{\mu\alpha\nu\beta}\left[(l_{3\xi}l_{2\beta}+l_{2\xi}l_{1\beta})\triangle_{\xi\alpha}+(l_{2\xi}l_{3\alpha}+l_{1\xi}l_{2\alpha})\triangle_{\xi\beta}\right] \tag{160}$$

and

$$p'_\nu T^{A\to AA}_{\lambda\mu\nu} = -2miT^{A\to AP}_{\lambda\mu} \\ -4\varepsilon_{\mu\alpha\nu\beta}\left[(k_{3\xi}k_{2\beta}+k_{1\xi}k_{3\beta})\triangle_{\xi\alpha}+(k_{2\xi}k_{3\alpha}+k_{3\xi}k_{1\alpha})\triangle_{\xi\beta}\right] \\ +4\varepsilon_{\mu\alpha\nu\beta}\left[(l_{1\xi}l_{3\beta}+l_{3\xi}l_{2\beta})\triangle_{\xi\alpha}+(l_{3\xi}l_{1\alpha}+l_{2\xi}l_{3\alpha})\triangle_{\xi\beta}\right]. \tag{161}$$

7 Final Analysis

We have established the Ward Identities, for fermions with spin 1/2 and equal masses to be satisfied by one, two and three point functions. We next explicitly calculated the one and two point functions and the ambiguous terms for the three point functions. The corresponding amplitudes, whose superficial degree of divergence, are cubic, quadratic or linear are shown to be potentially ambiguous. The Ward Identities which relate these amplitudes could then be violated.

In order to perform the calculations we followed a simple strategy concerning the divergencies; instead of adopting a specific regularization we simply have used general properties of an eventual regulating function whose momentum dependence should be even and a connection limit should exist. Therefore, in our results, are still contained those results corresponding to specific regularizations as we have shown. The conditions under which all the ambiguities can be avoided involve, in 4-D, the following objects:

$$\begin{aligned}
\Box_{\alpha\beta\mu\nu} &= \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{24k_\mu k_\nu k_\alpha k_\beta}{(k^2-m^2)^4} - g_{\alpha\beta}\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{4k_\mu k_\nu}{[(k^2-m^2)^3]} \\
&\quad - g_{\alpha\mu}\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{4k_\beta k_\nu}{[(k^2-m^2)^3]} - g_{\alpha\nu}\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{4k_\beta k_\mu}{[(k^2-m^2)^3]}, \qquad (162) \\
\nabla_{\mu\nu} &= \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{2k_\mu k_\nu}{(k^2-m^2)^2} - \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{g_{\mu\nu}}{(k^2-m^2)}, \qquad (163) \\
\triangle_{\mu\nu} &= \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{4k_\mu k_\nu}{(k^2-m^2)^3} - \int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{g_{\mu\nu}}{(k^2-m^2)^2}. \qquad (164)
\end{aligned}$$

The divergent part of the final results are expressed in terms of the three above relations plus $I_{log}(m^2)$ and $I_{quad}(m^2)$. In D.R. this objects can be obtained simultaneously as zero.

In the present context, the analysis of the results, regarding the ambiguities and symmetry relations, resulted rather transparent, once we look all conditions through the three forms above. Concerning this two aspects we can state what follows:

- **Ambiguities**

  In all amplitudes the dependence upon arbitrary choice of internal momenta of loops appear simply as <u>coefficients</u> for the differences between divergent integrals, eqs.(162)-(164). Given this fact the conclusion is immediate: all the ambiguities will be eliminated if they are simultaneously zero. Howeverit is not surprising that ambiguities can be eliminated if shifts were allowed i.e., provided we ignore the corresponding surface terms. Conversely, from this point of view, this conclusion can be looked upon as 4-D conditions to be satisfied by any regularization prescription which should have the consistency of D.R. whenever it applies [12]. However this is not the whole story.

- **Ward Identities**

  In our investigations we have found several examples in which ambiguities and symmetry violations are intimately connected and have the same origin. However there were also several instances in which Ward identities could be violated by unambiguous terms with the same structure of the relations eqs.(162)-(164). This is the case, for example, of $T_{\mu\nu}^{VV}$ and $T_{\mu\nu}^{AA}$ where the difference $\nabla_{\alpha\beta}$ appears with an unambiguous coefficient. Also the three point functions $T_{\mu\nu}^{SVV}$ and $T_{\mu\nu}^{SAA}$ involve the difference $\triangle_{\alpha\beta}$ as condition for the fulfillment of the W.I. The difference $\square_{\alpha\beta\mu\nu}$ would also appears as an unambiguous contribution for the four point function $T_{\mu\nu\lambda\rho}^{VVVVV}$ as can be easily checked.

We could then invert the analysis starting precisely by these amplitudes and extracting the conclusion that, independently of ambiguities, the objects $\square_{\alpha\beta\mu\nu}$, $\nabla_{\mu\nu}$ and $\triangle_{\mu\nu}$ should be obtained as zero. We would then, a posteriori, verify that these conditions eliminate all sources of ambiguities.

At this point we reach an important and rather surprising result: Following the strategy of Gerstein and Jackiw in ref.[1] to study W.I., which has been used historically to establish violations of symmetry relations, we found a set of conditions which allow all the W.I. to be satisfied. In this context the possibility of making use of ambiguities for any purpose is automatically eliminated. By imposing these referred conditions, the corresponding results of D.R. can be immediately mapped whenever it applies. Those corresponding Gerstein and Jackiw results, as we have shown in section 5, can be equally obtained from our results but not with the same interpretation for the objects $\square_{\alpha\beta\mu\nu}$, $\nabla_{\mu\nu}$ and $\triangle_{\mu\nu}$.

The situation is now the following: To establish or justify the existence of the anomaly phenomena, in the context of perturbative calculations, we need use a specific prescription to evaluate some divergent integrals. The traditional one, based on surface terms, could be used, in principle, to treat all the amplitudes for any theories but is discarded where the D.R. can be applied and accepted only for the treatment of pseudo-amplitudes where the D.R. cannot be used due its natural limitation. The two treatments lead to results that cannot be mapped one into another for places where both are applied.

If we are looking for an universal way to treat all divergent amplitudes in QFT, the above situation in unacceptable.

We arrive at two deeply different options: first if we adopt the interpretation corresponding the surface terms point of view we can get a picture for triangular anomalies that correspond to the one of the Gerstein an Jackiw, but, in consequence, we will plague all physical amplitudes with ambiguities and we loose the translational invariance, the main of the basics space-time symmetries, second if we adopt the D.R. interpretation for the objects $\Box_{\alpha\beta\mu\nu}$, $\nabla_{\mu\nu}$ and $\triangle_{\mu\nu}$ we have all symmetry relations satisfied, including those considered as anomalous.

This statement is an immediate consequence of our strategy in looking for perturbative calculations involving divergent amplitudes; the consistency conditions makes immaterial an eventual choice for the value of undefined quantities, because in all places of ocurrence they are multiplied by differences between divergent integrals of the same degree of divergence, that are identically zero.

Once we have concluded that there is no chance of consistency in calculations involving divergent integrals without the imposition of consistent conditions, a crucial question emerges: the combination of our treatment given to the divergent integral plus the strategy of Gerstein and Jackiw to verify Ward Identities in three point functions leads to the conclusion that there are no anomalies in triangular diagrams? The answer is no. The conclusion of our investigation, which at this point became transparent, is that this kind of analysis is not consistent, because we can find conditions that produce exactly the opposite conclusion of initial intentions.

What is then the correct procedure to discuss this problem? Our expectative resides on the explicitly calculations for the three point functions. The correct violation values for symmetry relations (anomalies) need to emerge in a natural way, free of ambiguities related to the arbitrary choice of internal labels. This is actually the essential point of the Sutherland paradox that states the impossibility to obtain simultaneously the three Ward Identities and the correct value in zero external momenta. In a calculation where one of these ingredients is absent we cannot extract conclusions about the anomaly phenomena. Work along these lines is presently under way and the preliminary results are in perfect agreement with the above arguments.

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